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Perfect Spline Solutions of L_∞ Extremal Problems by Control Methods

DONALD E. MCCLURE

Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912

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Methods of analyzing linear time-optimal control problems are adapted to the analysis of the extremal problem $\| Lf_0 \|_\infty = \inf_{f \in U} \| Lf \|_\infty$; L is a linear nth order differential operator and U is a flat in the Sobolev space $W^{n,\infty}[a, b]$. Existence and uniqueness of solutions are established for particular U determined by interpolation conditions at a and b. Solutions are characterized as perfect splines, enabling one to obtain solutions of perfect-spline interpolation problems. Further, existence of perfect-spline solutions is established for extremal and interpolation problems determined by more general flats U.

1. INTRODUCTION

Glaeser [3] sparked a current surge of interest in perfect-spline functions and related L_∞ extremal problems. The present paper is directly concerned with establishing results like those related in Glaeser's pioneering work, but with generalizations of the characterizing extremal problems such as those considered by Fisher and Jerome [1, 2]. We appeal to the strong connections between the extremal problems and problems of optimal control to draw on results and methods of analysis from control theory for the consideration of the L_∞ extremal problems and the characterization of their solutions as perfect splines. While control theory itself is a deep and rich area of analysis, the results and methods that we borrow from it are elementary ones. With these relatively simple methods we can still establish generalizations of certain basic and nontrivial results concerning perfect splines.

Glaeser's first results are related in a paper by Schoenberg [10]. A real-valued function $f(t)$ defined on the interval $[a, b]$ is a *polynomial spline function* of degree n if there are points $a = t_0 < t_1 < \cdots < t_k < t_{k+1} = b$ such that: (i) f is a polynomial of degree n on (t_j, t_{j+1}) for $j = 0,..., k$, and (ii) $f \in C^{n-1}[a, b]$. The points $t_k \in (a, b)$ are the *knots* of f. Such a function f is called a *perfect spline* if, in addition: (iii) $|f^{(n)}(t)|$ is constant on $[a, b]$ for $t \neq t_1,..., t_k$. There are two theorems of Glaeser with which we are principally concerned.

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THEOREM. *If 2n real values $x_0^{(\nu)}$, $x_1^{(\nu)}$ for $\nu = 1,..., n$ are specified, then the 2-point Hermite interpolation problem $f^{(\nu)}(a) = x_0^{(\nu+1)}$, $f^{(\nu)}(b) = x_1^{(\nu+1)}$ for $\nu = 0,..., n-1$ has a unique solution f_0 that is a perfect spline of degree n having fewer than n knots in (a, b).*

In relating this unique perfect spline f_0 to an extremal problem, Glaeser proved the following result.

THEOREM. *The perfect spline f_0 of the theorem above is the unique function that minimizes $\| f^{(n)} \|_\infty$ among all functions f for which $f^{(\nu)}(a) = x_0^{(\nu+1)}$, $f^{(\nu)}(b) = x_1^{(\nu+1)}$ for $\nu = 0,..., n-1$ and for which $f^{(\nu)}$ for $\nu = 0,..., n-1$ are absolutely continuous on $[a, b]$.*

Significant extensions of these early results were announced by Karlin [6]. Two of Karlin's results are in the same direction as Glaeser's theorems, but more general interpolation conditions than before are considered for the interpolation and extremal problems. The order of Karlin's results is the same as Glaeser's; first, existence of a perfect spline solution to the interpolation problem is stated and then this perfect spline is claimed to minimize $\| f^{(n)} \|_\infty$ among all f satisfying the imposed conditions. Further, the sharp bounds on the number of knots of an interpolating perfect spline carry over to the more general interpolation conditions. However, simple examples show that the solutions under Karlin's conditions need not be unique (see [1], for example).

The ways in which Fisher and Jerome have generalized both Glaeser's and Karlin's conditions on the extremal and interpolation problems determine the direction that we adopt here. First, in [1] they consider the problem of existence of a function f_0 that minimizes $\| Lf \|_\infty$ on an interval $[a, b]$, where: (i) L is a nonsingular linear differential operator of order n, (ii) functions f and their first $(n-1)$ derivatives are absolutely continuous on $[a, b]$, and (iii) f satisfies very general interpolation conditions at fixed points in the interval. These generalizations of the interpolation conditions are detailed in Section 4, where we relate our new results to extensions of those obtained by Fisher and Jerome. In [2], they show that "perfect-spline" solutions of the extremal problem exist, and thus obtain perfect-spline solutions of the interpolation problem related to the extremal problem. Perfect splines in this setting are functions f for which Lf has only a finite number of discontinuities in $[a, b]$ and $| Lf |$ is a constant a.e. The case $L = D^n$ corresponds to the problems considered by Glaeser and Karlin. Note that the order of presentation is reversed by Fisher and Jerome, first treating the extremal problem and then treating the perfect-spline interpolation problem. This direction is also exploited in the present analysis.

Schoenberg [10] has indicated relationships between perfect-spline problems and optimal control. Schoenberg redevelops a result of Louboutin [8] in giving an explicit expression for the interpolating perfect spline described in Glaeser's first theorem for the case $a = -1$, $b = 1$, $x_0^{(\nu)} = 0$ for $\nu = 1,\ldots, n$, $x_1^{(1)} = 1$ and $x_1^{(\nu)} = 0$ for $\nu = 2,\ldots, n$. Then appealing to the extremal property of this spline elicited in Glaeser's second theorem, Schoenberg uses it to construct a solution to a time-optimal control problem for a system governed by the operator D^n. Thus, Schoenberg brings analysis of perfect splines to bear on a problem in control. We reverse this process in drawing on methods from control to treat perfect splines and their interpolating and extremal properties.

The paper by Mangasarian and Schumaker [9] relates in spirit to the present one. They invoke results and methods from control theory for the analysis of L_p extremal problems, $1 < p < \infty$. In so doing, they convincingly demonstrate the power of this approach to extremal properties of splines and they anticipate the relevance of control methods to L_∞ extremal problems.

Problem formulations and main results are given in the next section. The proofs are developed in Section 3. The last section relates these results to an extension of earlier work by Fisher and Jerome [2].

2. Extremal Problems and Results

On a fixed finite interval $[a, b]$ of the line, consider functions f in the Sobolev space $W^{n,\infty}$; $W^{n,\infty} = \{f \in R^{[a,b]}: f^{(\nu)}$ is absolutely continuous for $\nu = 0,\ldots, n-1$ and $\|f^{(n)}\|_\infty$ is finite$\}$, where $\|\cdot\|_\infty$ is the essential sup norm.

Let $x_0^{(\nu)}$, $x_1^{(\nu)}$ for $\nu = 1,\ldots, n$ be $2n$ specified real values and define the subset U of $W^{n,\infty}$ by

$$U = \{f \in W^{n,\infty}: f^{(\nu)}(a) = x_0^{(\nu+1)}, f^{(\nu)}(b) = x_1^{(\nu+1)} \text{ for } \nu = 0,\ldots, n-1\}.$$

Let L denote a nonsingular nth order linear differential operator,

$$L = D^n + \sum_{\nu=1}^{n} a_\nu(t)\, D^{n-\nu},$$

where D is the operator of differentiation. We assume that the coefficient functions a_ν are in $C^{n-\nu}[a, b]$, so that L^*, the formal adjoint of L, also exists as a nonsingular nth order linear differential operator with continuous coefficients. Explicitly

$$(-1)^n L^* f = D^n f + \sum_{\nu=1}^{n} (-1)^\nu D^{n-\nu}(a_\nu f).$$

Consider L operating on U and define

$$\alpha_0 = \inf_{f \in U} \| Lf \|_\infty . \tag{2.1}$$

Our first concern is to establish the existence of f_0 in U for which the infimum in (2.1) is attained, i.e., for which

$$\| Lf_0 \|_\infty = \alpha_0 , \qquad f_0 \in U. \tag{2.2}$$

In the course of establishing the existence of a solution to Eq. (2.2), we also obtain characterizations in terms of perfect-spline properties.

We adopt standard extrapolations of notions of polynomial splines and perfect polynomial splines to define what is meant by "perfect spline" in the following discussion. In terms of the fixed nth order differential operator L, we refer to a real-valued function f on $[a, b]$ as a *spline function* of degree n if there are points $a = t_0 < t_1 < \cdots < t_k < t_{k+1} = b$ such that

(i) $f \in W^{n,\infty}$, and

(ii) Lf assumes a constant value u_j on each interval (t_j, t_{j+1}) for $j = 0, 1,..., k$. Such points t_j in (a, b) are called *knots* of f. A spline f is called a *perfect spline* if, in addition,

(iii) $| Lf |$ is constant on $[a, b] - \{t_j\}_{j=1}^{k}$.

For some of the characterizations of solutions to (2.2) we impose an additional assumption concerning the operator L and call this Property T. The operator L^* is said to possess Property T (disconjugacy) if its null space is spanned by a Tchebycheff system on $[a, b]$; this means that a nontrivial solution ϕ of $L^*\phi = 0$ on $[a, b]$ has at most $n - 1$ zeros in $[a, b]$.

The first result concerns the extremal problem (2.2).

THEOREM 1. *There exists a unique function f_0 in U that satisfies* (2.2). *The function f_0 is a perfect spline on $[a, b]$; that is, there are $k \geqslant 0$ interior points $t_1 < t_2 < \cdots < t_k$ in (a, b) such that $Lf_0(t)$ exists for all t in $[a, b] - \{t_j\}_{j=1}^{k}$ and $| Lf_0 | = \alpha_0$ excepting the points t_j. Further, if L^* possesses Property T on $[a, b]$ then $k \leqslant n - 1$.*

Much of the statement of Theorem 1 is included in the previous results of Fisher and Jerome [2]. It is proven anew in Section 3, since it is an easy by-product of the methods that lead to new results. In addition to Theorem 1 we prove the following theorem.

THEOREM 2. *If f in U is a perfect spline with interior knots $t_1 < \cdots < t_k$ in (a, b) and $| Lf | = \alpha > \alpha_0$, where α_0 is given by* (2.1), *and if L^* possesses Property T on $[a, b]$ then $k \geqslant n$.*

Theorems 1 and 2 yield an analogue of Glaeser's first theorem concerning existence and uniqueness of interpolating perfect splines.

THEOREM 3. *If L^* possesses Property T on $[a, b]$ then there exists a unique perfect spline f_0 in U with fewer than n knots in (a, b).*

Finally we address the question of existence of perfect splines in U such as described in Theorem 2 and we obtain the result that allows the extensions of earlier work by Fisher and Jerome described in Section 4.

THEOREM 4. *If $\alpha > \alpha_0$ (2.1), then there exists a perfect spline f in U with $| Lf | \equiv \alpha$ except at the knots of f. If L^* possesses Property T on $[a, b]$ then such a perfect spline f has at least n knots in (a, b), and there exist exactly two perfect splines in U that have precisely n knots in (a, b). These two splines are distinguished by the sign of Lf near a.*

The same brand of analysis is used to prove each of these theorems. The first step is a reformulation of the extremal problem (2.2) in control terms.

3. CONTROL FORMULATION AND PROOFS

The notation and definitions of Section 2 are used in this section.

Consider the differential equation

$$Lf = u, \qquad u \in L_\infty[a, b]. \tag{3.1}$$

Any solution of (3.1) is a member of $W^{n,\infty}$ and, conversely, any function in $W^{n,\infty}$ satisfies an equation like (3.1). To relate solutions of (3.1) to the subset U of $W^{n,\infty}$, impose the initial conditions

$$f^{(\nu)}(a) = x_0^{(\nu+1)}, \qquad \text{for} \qquad \nu = 0,..., n-1, \tag{3.2}$$

where the $x_0^{(\nu)}$ are the specified real-values used to describe U. The question of existence of a solution f_0 of the extremal problem (2.2) can be expressed equivalently as a question of existence of a function (control) u_0 in $L_\infty[a, b]$ for which $\| u_0 \|_\infty = \alpha_0$ and for which the associated solution f_0 of (3.1) and (3.2) satisfies

$$f^{(\nu)}(b) = x_1^{(\nu+1)}, \qquad \text{for} \qquad \nu = 0,..., n-1. \tag{3.3}$$

We refer to the function u in (3.1) as a control.

The control problem is more readily analyzed in terms of systems of equations equivalent to (3.1)–(3.3). Let $A(t)$ be an $n \times n$ matrix

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & & \cdots & 1 \\ -a_n(t) & -a_{n-1}(t) & & \cdots & -a_1(t) \end{bmatrix}$$

and let B be the column n-vector

$$B = \text{col}(0, 0, \ldots, 0, 1).$$

The system

$$X' = A(t)\, X + Bu, \qquad u \in L_\infty[a, b] \tag{3.4}$$

describing an n-vector function $X(t) = \text{col}(x_1(t), \ldots, x_n(t))$ is equivalent to (3.1) with the identification $f^{(\nu)} = x_{\nu+1}$, for $\nu = 0, \ldots, n-1$. Denote

$$X_0 = \text{col}(x_0^{(1)}, \ldots, x_0^{(n)}) \text{ and } X_1 = \text{col}(x_1^{(1)}, \ldots, x_1^{(n)})$$

where the $x_0^{(\nu)}$ and $x_1^{(\nu)}$ are the same values that define U. The initial condition (3.2) associated with (3.1) translates into

$$X(a) = X_0 \tag{3.5}$$

associated with system (3.4). In the same way, the condition (3.3) on the state we seek to attain is expressed as

$$X(b) = X_1. \tag{3.6}$$

Eqs. (3.4)–(3.6) are equivalent to (3.1)–(3.3) and problem (2.2) may thus be expressed as follows:

PROBLEM. Does there exist a control u_0 in $L_\infty[a, b]$ with $\| u_0 \|_\infty = \alpha_0$ so that the solution of (3.4) and (3.5) for $u = u_0$ satisfies (3.6)?

Our affirmative analysis of this question will yield characterizations of such u_0 that we can turn into statements about the perfect-spline nature of solutions f_0 of (2.2). How this will work is probably evident to readers familiar with fundamentals of control theory. Perfect splines are associated with controls u of constant absolute value by (3.1), that is, with the so-called Bang-Bang controls. The existence questions of Theorems 1 and 4 translate into questions of attainable states for Bang-Bang controls, and appeal to the Bang-Bang Principle already yields partial answers to the concerns of these theorems (see Hermes and LaSalle [5]). The proofs of this section are effectively verifications of this basic principle of control for the system (3.1).

Now in seeking a control u_0 for which $\| u_0 \|_\infty = \alpha_0$ we are imposing a power constraint on the control in (3.4). It is convenient to define two classes of *admissible controls*, where power constraints are imposed. Define, for any $\alpha \geqslant 0$

$$\Omega_\alpha = \{u \in L_\infty[a, b]: \| u \|_\infty \leqslant \alpha\}$$

and

$$\Omega_\alpha{}^0 = \{u \in L_\infty[a, b]: | u(t)| = \alpha \text{ a.e. in } [a, b]\}.$$

Special interest centers on the classes $\Omega_\alpha{}^0$ since perfect splines are associated with controls in this class.

Solutions of (3.4) and (3.5) are conveniently expressed by the variation of constants formula in terms of a fundamental matrix solution Ψ of the homogeneous system $X' = A(t) X$. X is a solution of (3.4) and (3.5) if and only if

$$X(t) = \Psi(t)\, \Psi^{-1}(a)\, X_0 + \Psi(t) \int_a^t \Psi^{-1}(s)\, Bu(s)\, ds. \tag{3.7}$$

To describe attainable states at time b, which X_1 should be, define the *attainable sets* $\mathcal{A}_\alpha$ and $\mathcal{A}_\alpha{}^0$, which are images of Ω_α and $\Omega_\alpha{}^0$ in R^n under the mapping (3.7). Let

$$\mathcal{A}_\alpha = \{X(b) \in R^n: X(\ ) \text{ satisfies (3.7) on } [a, b] \text{ for some } u \in \Omega_\alpha\} \tag{3.8}$$

and

$$\mathcal{A}_\alpha{}^0 = \{X(b) \in R^n: X(\ ) \text{ satisfies (3.7) on } [a, b] \text{ for some } u \in \Omega_\alpha{}^0\}.$$

The Bang-Bang Principle, to which we alluded, states $\mathcal{A}_\alpha = \mathcal{A}_\alpha{}^0$. Theorem 1 is proven by showing $X_1 \in \mathcal{A}_{\alpha_0}^0$ and by invoking necessary conditions on the control $u_0 \in \Omega_{\alpha_0}^0$ that drives the system to X_1.

We first establish useful properties of the sets $\mathcal{A}_\alpha$. All of these properties are either proven in [5] or are direct consequences of results established therein. Brief arguments are presented here in the interest of completeness.

LEMMA 1. *The subsets $\mathcal{A}_\alpha$ of R^n given by* (3.8) *have the following properties*:

(i) *$\mathcal{A}_\alpha$ is a compact convex subset of R^n for all $\alpha \geqslant 0$*;

(ii) *X is a boundary point of $\mathcal{A}_\alpha$ if and only if there is a nonzero vector η such that*

$$X = \Psi(b)\, \Psi^{-1}(a)\, X_0 + \int_a^b \Psi(b)\, \Psi^{-1}(s)\, Bu^*(s)\, ds \tag{3.9}$$

and

$$u^*(s) = \alpha \operatorname{sgn}[\eta^t \Psi(b)\, \Psi^{-1}(s)\, B]; \tag{3.10}$$

(iii) *if* $\alpha > 0$, *then the interior of* $\mathcal{O}_\alpha$ (int $\mathcal{O}_\alpha$) *is nonempty*;

(iv) *if* $0 \leqslant \alpha < \beta$, *then* $\mathcal{O}_\alpha \subseteq \text{int } \mathcal{O}_\beta$;

(v) *with the Hausdorff metric on compact subsets of* R^n, *the mapping* $\alpha \to \mathcal{O}_\alpha$ *is a continuous set-valued function on* $[0, \infty)$;

(vi) *if* $\alpha > 0$ *and* $X \in \text{int } \mathcal{O}_\alpha$, *then for some positive* δ, $X \in \text{int } \mathcal{O}_{\alpha-\delta}$.

Remark. In (3.10) η^t denotes the transpose of the vector η and sgn ( ) denotes the signum function. In the present analysis we need not be concerned with its indeterminate value for a zero argument.

Proof. (i) Ω_α is convex and the integral expression in the representation (3.7) is linear in u. Thus, $\mathcal{O}_\alpha$ is convex. Also, Ω_α is a norm-closed ball in $L_\infty[a, b]$, so it is compact in the weak* topology of $L_\infty[a, b]$. Eq. (3.7), with $t = b$, describes a continuous transformation from $L_\infty[a, b]$ to R^n in the weak* topology, so the compactness of $\mathcal{O}_\alpha$ follows from that of Ω_α .

(ii) Suppose X is a boundary point of $\mathcal{O}_\alpha$. Since $\mathcal{O}_\alpha$ is compact and convex it has a support plane passing through X [5, p. 35]. There is a nonzero vector η such that $\eta^t(X - Y) \geqslant 0$ for all $Y \in \mathcal{O}_\alpha$. Let u^* denote a control in Ω_α associated with X (3.9) and let u be arbitrary in Ω_α . Using (3.7), the inequality $\eta^t(X - Y) \geqslant 0$ implies

$$\int_a^b [\eta^t\Psi(b)\,\Psi^{-1}(s)\,B]\,u^*(s)\,ds \geqslant \int_a^b [\eta^t\Psi(b)\,\Psi^{-1}(s)\,B]\,u(s)\,ds$$

for all $u \in \Omega_\alpha$. This yields the necessary representation (3.10) for $u^* \in \Omega_\alpha$.

Before establishing sufficiency of (3.9) and (3.10), we observe that (3.10) with $\eta \neq 0$ essentially determines a unique control. The matrix $\Psi(b)\,\Psi^{-1}(s)$ as a function of s is a fundamental matrix solution of the adjoint system $X' = -XA(s)$ interpreted as an equation in the row-vector X (see [4]). Since the system (3.4) is equivalent to the scalar equation (3.1), the homogeneous adjoint system $X' = -XA(s)$ is equivalent to the adjoint equation $L^*\phi = 0$, with appropriate identification of solutions. In particular, the last column of $\Psi(b)\,\Psi^{-1}(s)$, which is given by $\Psi(b)\,\Psi^{-1}(s)\,B$, contains n linearly independent solutions of $L^*\phi = 0$. Thus, $\eta^t\Psi(b)\,\Psi^{-1}(s)\,B$ is just a nontrivial solution of $L^*\phi = 0$, for any nonzero vector η. With the assumptions we have imposed on L^*, that it be a nonsingular nth order linear differential operator with continuous coefficients on $[a, b]$, nontrivial solutions $\eta^t\Psi(b)\,\Psi^{-1}(s)\,B$ of $L^*\phi = 0$ can only have a finite number of isolated zeros in $[a, b]$. Only at these points is $\text{sgn}[\eta^t\Psi(b)\,\Psi^{-1}(s)\,B]$ indeterminate. So, as claimed, u^* is essentially uniquely determined by (3.10).

Now suppose $X \in \mathcal{O}_\alpha$ has a representation, (3.9) and (3.10), for some nonzero vector η. Then u^* maximizes $\int_a^b [\eta^t\Psi(b)\,\Psi^{-1}(s)\,B]\,u(s)\,ds$ with respect to u in Ω_α . Invoking (3.7) again, we obtain $\eta^t(X - Y) \geqslant 0$ for all Y in $\mathcal{O}_\alpha$.

Thus η determines a support plane of $\mathcal{O}_\alpha$, passing through the point X. So X must be a boundary point of $\mathcal{O}_\alpha$.

(iii) Let $\alpha > 0$ and let $u \in \Omega_\alpha - \Omega_\alpha^0$. Consider $X \in \mathcal{O}_\alpha$ given by

$$X = \Psi(b)\, \Psi^{-1}(a)\, X_0 + \int_a^b \Psi(b)\, \Psi^{-1}(s)\, Bu(s)\, ds.$$

If X were a boundary point of $\mathcal{O}_\alpha$, its associated control would be uniquely determined by an expression (3.10), following the argument of part (ii). But since $u \notin \Omega_\alpha^0$, u cannot be expressed in this form. So X must be an interior point of $\mathcal{O}_\alpha$.

(iv) Let $0 \leqslant \alpha < \beta$. Since $\Omega_\alpha \subseteq \Omega_\beta$, clearly $\mathcal{O}_\alpha \subseteq \mathcal{O}_\beta$. Also, by the argument of (iii), if $X \in \mathcal{O}_\alpha$ is associated with control u, then $u \in \Omega_\beta - \Omega_\beta^0$ so $X \in \operatorname{int} \mathcal{O}_\beta$.

(v) Let $\epsilon > 0$, and let $0 \leqslant \alpha < \beta$. Denote $\delta = \beta - \alpha$. It suffices to show that $\mathcal{O}_\beta$ is contained in an ϵ-neighborhood of $\mathcal{O}_\alpha$ when δ is sufficiently small. $\mathcal{O}_\alpha$ is always contained in an ϵ-neighborhood of $\mathcal{O}_\beta$, by (iv). Consider X in $\mathcal{O}_\beta$ associated with control u_β in Ω_β. Define u_α in Ω_α by $u_\alpha(s) = u_\beta(s)$ when $| u_\beta(s)| \leqslant \alpha$, and $u_\alpha(s) = \alpha \operatorname{sgn} u_\beta(s)$ when $| u_\beta(s)| > \alpha$. By this construction $\| u_\beta - u_\alpha \| \leqslant \delta$. Let Z be the point in $\mathcal{O}_\alpha$ associated with control u_α. Using (3.7) to represent X and Z, we obtain a bound on the Euclidean distance $\| X - Z \|$ between X and Z in R^n;

$$\begin{aligned} \| X - Z \| &\leqslant \int_a^b \| \Psi(b)\, \Psi^{-1}(s)\, B \|\, | u_\beta(s) - u_\alpha(s)|\, ds \\ &\leqslant \delta \int_a^b \| \Psi(b)\, \Psi^{-1}(s)\, B \|\, ds. \end{aligned}$$

The bound is independent of the point X in $\mathcal{O}_\beta$, and it can be made less than ϵ by choosing δ small.

(vi) Let $\alpha > 0$ and suppose $X \in \operatorname{int} \mathcal{O}_\alpha$. Fix $\epsilon > 0$ such that

$$\{Z : \| Z - X \| < \epsilon\} \subseteq \mathcal{O}_\alpha .$$

By (v), we can fix a positive δ such that the Hausdorff distance between $\mathcal{O}_\alpha$ and $\mathcal{O}_{\alpha-\delta}$ is less than $\epsilon/2$. We claim that $\{Z: \| Z - X \| < \epsilon/2\} \subseteq \mathcal{O}_{\alpha-\delta}$. Suppose not and let Y be a fixed point in $\{Z: \| Z - X \| < \epsilon/2\} - \mathcal{O}_{\alpha-\delta}$. Since $\mathcal{O}_{\alpha-\delta}$ is closed and convex there is hyperplane passing through Y that does not intersect $\mathcal{O}_{\alpha-\delta}$, i.e., there is a unit vector η such that $\eta^t(Y - W) > 0$ for all W in $\mathcal{O}_{\alpha-\delta}$. Now consider the point $Y^* = Y + (\epsilon/2)\, \eta$. We have $\| X - Y^* \| \leqslant \| X - Y \| + \| Y - Y^* \| < \epsilon$, implying $Y^* \in \mathcal{O}_\alpha$. Further, for any W in $\mathcal{O}_{\alpha-\delta}$, $\| Y^* - W \|^2 = \|(\epsilon/2)\, \eta + Y - W \|^2 = (\epsilon^2/4) + \| Y - W \|^2$

$+ \epsilon\eta^t(Y - W) > (\epsilon/2)^2$. Thus $Y^* \in \mathcal{O}l_\alpha$ and $\| Y^* - W\| > \epsilon/2$ for all $W \in \mathcal{O}l_{\alpha-\delta}$, which contradicts the choice of δ. Thus $X \in \text{int } \mathcal{O}l_{\alpha-\delta}$.

Lemma 1 is proved. Theorem 1, stated in Section 2 is now easily proven.

Proof of Theorem 1. Let $\alpha_0 = \inf_{f\in U} \| Lf\|_\infty$ (2.1). By construction of the system, (3.4) and (3.5), and its association with the extremal problem (2.2), $\alpha_0 = \inf\{\alpha \geqslant 0 : X_1 \in \mathcal{O}l_\alpha\}$. By (iv) of Lemma 1, $X_1 \in \mathcal{O}l_\alpha$ for all $\alpha > \alpha_0$. Since $\mathcal{O}l_\alpha$ depends continuously on α and $\mathcal{O}l_{\alpha_0}$ is closed this implies $X_1 \in \mathcal{O}l_{\alpha_0}$. Suppose $\alpha_0 > 0$; when $\alpha_0 = 0$ the conclusions of the theorem are obvious from the fact $X_1 \in \mathcal{O}l_{\alpha_0}$.

Now X_1 cannot be an interior point of $\mathcal{O}l_{\alpha_0}$, or else by (vi) $X_1 \in \mathcal{O}l_{\alpha_0-\delta}$ for some positive δ, contradicting the definition of α_0. Thus X_1 is a boundary point of $\mathcal{O}l_{\alpha_0}$. By (ii) of Lemma 1 and its proof, the control u^* that attains X_1 is uniquely determined by (3.10). The associated solution $X(t)$ of (3.4)–(3.6) exists and is unique. The associated solution f_0 of (3.1)–(3.3) with $u = u^*$ is in U, it satisfies (2.2) and it is unique.

The characterization of f_0 follows from the properties of u^* given by (3.10). By the remarks on the proof of (ii) in Lemma 1, $u^*(s) = \alpha_0 \operatorname{sgn}[\phi(s)]$, where ϕ is a nontrivial solution of $L^*\phi = 0$. Thus, there are a finite number of points $t_1 < t_2 < \cdots < t_k$ in (a, b), $k \geqslant 0$, where u^* changes sign. These are zeros of ϕ. Excepting these points, i.e., for $t \in [a, b] - \{t_j\}_{j=1}^k$, $Lf_0(t) = u^*(t)$ exists and $| Lf_0(t)| = \alpha_0$. Finally, if L^* possesses Property T on $[a, b]$, then ϕ can have at most $n - 1$ zeros, so $k \leqslant n - 1$. This completes the proof.

Theorem 2 follows as readily from properties observed in Lemma 1.

Proof of Theorem 2. Suppose f in U is a perfect spline with knots

$$t_1 < \cdots < t_k$$

in (a, b) and $| Lf | = \alpha \geqslant \alpha_0$. Suppose $k \leqslant n - 1$. If L^* possesses Property T on $[a, b]$ then there is a function ϕ satisfying $L^*\phi = 0$ such that ϕ changes sign at each point t_j in (a, b) for $j = 1,..., k$ and ϕ does not change sign at any other point in (a, b) (see [7, p. 30]). By the argument of Lemma 1(ii) there is a nonzero vector η such that $\phi(s) = \eta^t\Psi(b)\, \Psi^{-1}(s)\, B$. Since ϕ changes sign exactly at the points where Lf does, we obtain the representation

$$Lf(s) = \alpha \operatorname{sgn}[\pm\eta^t\Psi(b)\, \Psi^{-1}(s)\, B].$$

By (ii) of Lemma 1, this implies X_1 is a boundary point of $\mathcal{O}l_\alpha$. In turn, (iv) implies $\alpha = \alpha_0$ and the theorem is proved.

Theorem 3 is an immediate consequence of Theorems 1 and 2, so we can omit a more detailed argument.

We adopt a constructive approach to obtain the results of Theorem 4. With $\alpha > \alpha_0$, we can already say from the preceding analysis that $X_1 \in \mathcal{O}l_\alpha$.

The Bang-Bang Principle from control says then that $X_1 \in \mathcal{O}\!\mathcal{l}_\alpha{}^0$, which is part of a conclusion that we want to draw. However, the Bang-Bang Principle does not tell us about regularity properties of a Bang-Bang control u in $\Omega_\alpha{}^0$ that will attain the state X_1. The constructive approach we follow will yield this kind of information.

The construction of a perfect spline f in U, with $|Lf(t)| = \alpha$ except at knots starts by considering solutions X of (3.4) and (3.5) for which $u(t) \equiv \alpha$ in an interval to the right of $t = a$. We could as well start by fixing $u(t) \equiv -\alpha$ to the right of a, and the alternative construction would yield perfect splines distinct from those obtained by the route adopted. This remark is at the root of the last two statements of Theorem 4.

Fix $\alpha > 0$ and let Y be the unique solution on $[a, b]$ of

$$Y' = A(t)\, Y + B\alpha; \qquad Y(a) = X_0 . \tag{3.11}$$

Associated with Y, we define attainable sets $\mathcal{O}\!\mathcal{l}(t)$ that describe points in R^n attainable at time b starting from $Y(t)$ at time t and applying a control from Ω_α to (3.4):

$$\begin{aligned} \mathcal{O}\!\mathcal{l}(t) = \{&X(b) \in R^n\colon X(\ ) \text{ satisfies (3.4) on } [t, b], \\ &X(t) = Y(t), \text{ and } u \in \Omega_\alpha\} \text{ for } a \leqslant t \leqslant b. \end{aligned} \tag{3.12}$$

Properties of the set-valued function $\mathcal{O}\!\mathcal{l}(t)$ and its values are described in the following lemma.

LEMMA 2. *The set-valued function* $\mathcal{O}\!\mathcal{l}(t)$ *on* $[a, b]$ *defined by* (3.11) *and* (3.12) *has the following properties*:

(i) $\mathcal{O}\!\mathcal{l}(a) = \mathcal{O}\!\mathcal{l}_\alpha$ [Eq. (3.8)];

(ii) *for each t in* $[a, b]$, $\mathcal{O}\!\mathcal{l}(t)$ *is a compact convex subset of* R^n;

(iii) *if* $a \leqslant t \leqslant s \leqslant b$, *then* $\mathcal{O}\!\mathcal{l}(s) \subseteq \mathcal{O}\!\mathcal{l}(t)$;

(iv) *with the Hausdorff metric on compact subsets of* R^n, $\mathcal{O}\!\mathcal{l}(t)$ *is a continuous function on* $[a, b]$;

(v) *if* $t < b$, *then int* $\mathcal{O}\!\mathcal{l}(t)$ *is nonempty*;

(vi) *if* $t < b$ *and* $X \in$ *int* $\mathcal{O}\!\mathcal{l}(t)$, *then, for some positive* δ, $X \in \operatorname{int} \mathcal{O}\!\mathcal{l}(t + \delta)$.

Proof. Except for property (iv), all parts are proved by arguments completely analogous to those used for the corresponding parts of Lemma 1. Therefore we only present the argument for part (iv).

Let $\epsilon > 0$ and let $a \leqslant t < s \leqslant b$. Denote $\delta = s - t$. It suffices to show that $\mathcal{O}\!\mathcal{l}(t)$ is contained in an ϵ-neighborhood of $\mathcal{O}\!\mathcal{l}(s)$ when δ is sufficiently small. Consider X in $\mathcal{O}\!\ell(t)$ associated with control u on $[t, b]$. Define u^* by $u^*(\tau) = \alpha$ for $t \leqslant \tau < s$ and $u^*(\tau) = u(\tau)$ for $s \leqslant \tau \leqslant b$. Let Z be the point in $\mathcal{O}\!\ell(t)$ associated with control u^*. Also $Z \in \mathcal{O}\!\ell(s)$, since $u^*(\tau) = \alpha$ on $[t, s)$. Using (3.7) to represent X and Z, we can bound the distance $\| X - Z \|$;

$$\| X - Z \| \leqslant \int_t^s \| \Psi(b) \Psi^{-1}(\tau) B \| \, | u(\tau) - u^*(\tau)| \, d\tau \leqslant k \cdot \delta.$$

The integral in the bound collapses to the interval $[t, s)$ since u and u^* agree elsewhere; the constant k in the second part of the bound can be chosen independent of X in $\mathcal{O}\!\ell(t)$, since $| u(\tau) - u^*(\tau)| \leqslant 2\alpha$ and since

$$\| \Psi(b) \Psi^{-1}(\tau) B \|$$

is uniformly bounded on $[a, b]$. Uniform continuity of $\mathcal{O}\!\ell(t)$ on $[a, b]$ follows immediately from this bound.

In passing we note that the characterization of boundary points of $\mathcal{O}\!\ell_\alpha$ in Lemma 1 carries over to an analogous characterization of boundary points of $\mathcal{O}\!\ell(t)$. All that is changed is the interval over which control is not fixed. $\mathcal{O}\!\ell(t)$ is to the interval $[t, b]$ as $\mathcal{O}\!\ell_\alpha$ is to the interval $[a, b]$.

Lemma 2 provides the tools for the proof of Theorem 4.

Proof of Theorem 4. Fix $\alpha > \alpha_0$. By Theorem 1, $X_1 \in \mathcal{O}\!\ell_{\alpha_0}$ and by Lemma 1 (iv), $X_1 \in \operatorname{int} \mathcal{O}\!\ell_\alpha$. By Lemma 2 (i) and (vi), $X_1 \in \operatorname{int} \mathcal{O}\!\ell(t)$ for t in a neighborhood of a. Define

$$t_1 = \operatorname{lub} \{t \in [a, b] : X_1 \in \operatorname{int} \mathcal{O}\!\ell(t)\}.$$

Clearly $a < t_1 \leqslant b$.

We claim that X_1 is a boundary point of $\mathcal{O}\!\ell(t_1)$. From the definition of t_1 and Lemma 2(iii), $X_1 \in \mathcal{O}\!\ell(t)$ for $a \leqslant t < t_1$. By (iv) and since $\mathcal{O}\!\ell(t_1)$ is closed, therefore $X_1 \in \mathcal{O}\!\ell(t_1)$. If X_1 were an interior point of $\mathcal{O}\!\ell(t_1)$, then by (vi), X_1 would be an interior point of $\mathcal{O}\!\ell(t_1 + \delta)$ for some positive δ, contradicting the definition of t_1. Thus, X_1 is a boundary point of $\mathcal{O}\!\ell(t_1)$.

Suppose $t_1 < b$. If L^* possesses Property T this is necessarily true, since otherwise $Y(\ )$ would determine a solution of the interpolation problem with no knots in (a, b), contradicting Theorem 2.

Now use Lemma 1(ii) and the remark that relates the sets $\mathcal{O}\!\ell(t)$ to the sets $\mathcal{O}\!\ell_\alpha$ of Lemma 1. Since X_1 is a boundary point of $\mathcal{O}\!\ell(t_1)$, there is a unique control u on $[t_1, b]$ of the form

$$u(s) = \alpha \operatorname{sgn}[\eta^t \Psi(b) \Psi^{-1}(s) B], \qquad t_1 \leqslant s \leqslant b,$$

where η is a nonzero vector, such that

$$X_1 = \Psi(b)\,\Psi^{-1}(t_1)\,Y(t_1) + \int_{t_1}^{b} \Psi(b)\,\Psi^{-1}(s)\,Bu(s)\,ds.$$

Define $u^*(s) = \alpha$ for $a \leqslant s < t_1$ and $u^*(s) = u(s)$ for $t_1 \leqslant s \leqslant b$. The first component f of the solution X on $[a, b]$ of $X' = AX + Bu^*$, $X(a) = X_0$, is a perfect spline on $[a, b]$ satisfying the interpolation conditions at a and b. The function f has no knots in (a, t_1), and it has a finite number of knots in (t_1, b) coinciding with the isolated points where $\eta^t\Psi(b)\,\Psi^{-1}(s)\,B$ changes sign. Further, by the construction, $|Lf| \equiv \alpha$ except at the knots.

If L^* possesses Property T on $[a, b]$, then by Theorem 2, f must have at least n knots in (a, b). From the form of $u^*(s)$ on (t_1, b) and the argument of Lemma 1(ii), f has at most $n - 1$ knots in (t_1, b). Since f has no knots in (a, t_1) it must have a knot at t_1 and precisely $n - 1$ knots in (t_1, b). Thus f has exactly n knots in (a, b).

Up to a sign change in Lf in a neighborhood of $t = a$, f is unique in this regard. For suppose g is a perfect spline satisfying $|Lg| \equiv \alpha$ and fitting the interpolation conditions. Let s_1 denote the smallest knot of g in (a, b), and suppose $Lg = \alpha$ on (a, s_1). If $s_1 < t_1$, then $X_1 \in \operatorname{int} \mathscr{A}(s_1)$ and by the argument of Theorem 2, g would necessarily have at least n knots in (s_1, b). Adding the knot at s_1, g necessarily has at least $n + 1$ knots in (a, b). On the other hand, if $s_1 > t_1$ we obtain a contradiction to the uniqueness of $u(s)$ defined above on $[t_1, b]$. The uniqueness of $u(s)$ dictates that a perfect spline satisfying the conditions imposed on g must have a knot in $(a, t_1]$, since otherwise there would be two distinct controls on $[t_1, b]$ satisfying $|u(s)| \leqslant \alpha$ and attaining state X_1 at b from $Y(t_1)$ at t_1. Thus if g has exactly n knots in (a, b) its first knot must by t_1; it agrees with f on $[a, t_1]$ and the uniqueness of the control u on $[t_1, b]$ establishes the uniqueness of f.

Similarly, we can construct a unique perfect spline with exactly n knots in (a, b), satisfying the interpolation conditions and $|Lf| \equiv \alpha$, and conditioned by $Lf(t) = -\alpha$ in a neighborhood of a. The theorem is proved.

4. Results of Fisher and Jerome

Theorem 4 readily applies to a generalization of the main result of Fisher and Jerome [2]. We briefly describe their problem and results.

Extremal problems like (2.1) and (2.2) related to a nonsingular nth order linear differential operator are considered. However, much more general interpolation conditions are considered. We adopt the assumptions of Section 2 regarding the regularity of the coefficients $a_\nu(\ )$ of L.

To describe the interpolation conditions, let $a = x_1 < x_2 < \cdots < x_m = b$ be m points in $[a, b]$. Associated with each of the points x_i, introduce linear functionals L_{ij} on $W^{n,\infty}$ defined by

$$L_{ij}f = \sum_{\nu=0}^{n-1} a_{ij}^{(\nu)} D^\nu f(x_i), \qquad \text{for} \qquad j = 1,\ldots, k_i \text{ and } i = 1,\ldots, m;$$

the $a_{ij}^{(\nu)}$ denote prescribed real values such that, for each fixed i, the k_i n-tuples $(a_{ij}^{(0)},\ldots, a_{ij}^{(n-1)})$, $1 \leqslant j \leqslant k_i$, are linearly independent, and each k_i satisfies $1 \leqslant k_i \leqslant n$. Let r_{ij} be prescribed real numbers and define the class U^* by

$$U^* = \{f \in W^{n,\infty}: L_{ij}f = r_{ij}, \text{ for } 1 \leqslant j \leqslant k_i \text{ and } 1 \leqslant i \leqslant m\}.$$

The linear independence assumption on the functionals L_{ij} assures that U^* is nonempty.

Consider L operating on U^* and define

$$\alpha_0^* = \inf_{f\in U^*} \| Lf \|_\infty . \tag{4.1}$$

The first concern is to establish existence of f_0 in U^* for which the infimum in (4.1) is attained i.e., for which

$$\| Lf_0 \|_\infty = \alpha_0^*, \qquad f_0 \in U^*. \tag{4.2}$$

The first theorem of [1] addresses this question.

THEOREM 5. *The minimization problem,* (4.1) *and* (4.2), *has a solution g in $W^{n,\infty}$ and the class $S(U^*)$ of all such solutions g for a fixed choice of U^* is a convex set. Let $S_1(U^*) = S(U^*)$ and, for $2 \leqslant i \leqslant m$, let $S_i(U^*)$ consist of all solutions to the minimization problem*

$$\alpha_{i-1} = \inf\{\| Lg \|_{L_\infty(x_{i-1},x_i)}: g \in S_{i-1}(U^*)\}.$$

Then each $S_i(U^*)$ is nonempty; in particular, $S_m(U^*) = \bigcap_{i=1}^m S_i(U^*)$ is nonempty.

(Notation $\| \ \|_{L_\infty(x_{i-1},x_i)}$ is adopted to make clear the interval to which the norm is restricted.) The successive construction of the classes $S_i(U^*)$ produces solutions of (4.2) that are "locally optimal" on subintervals (x_{i-1}, x_i). The characterization of solutions obtained by this construction follows additional assumptions on the operator L and the functionals L_{ij}.

Regarding L, Fisher and Jerome assume

(I) $a_\nu \in C^{n-\nu}[a, b]$ and L^* possesses Property T on $[a, b]$.

A uniqueness characterization of solutions to the extremal problem relies on the further assumption concerning the functionals L_{ij}. Let n_0 be the largest positive integer with the property that for any n_0 consecutive points among $x_1, \ldots, x_m$ the sum of the associated integers k_i does not exceed n. Necessarily, $1 \leqslant n_0 \leqslant n$. Regarding the L_{ij}, assume

(II) (a) For every n_0 consecutive points $x_\lambda, \ldots, x_{\lambda+n_0-1}$ and prescribed values y_{ij} there is a function ϕ in the null space of L satisfying $L_{ij}\phi = y_{ij}$ for $1 \leqslant j \leqslant k_i$ and $\lambda \leqslant i \leqslant \lambda + n_0 - 1$;

(b) for any $n_0 + 1$ consecutive points $x_\lambda, \ldots, x_{\lambda+n_0}$ for which $\sum_{i=\lambda}^{\lambda+n_0} k_i \geqslant n + 1$ the equations $L_{ij}\phi = 0$ for $1 \leqslant j \leqslant k_i$ and $\lambda \leqslant i \leqslant \lambda + n_0$ and ϕ in the null space of L imply $\phi \equiv 0$.

The following theorem is the principal characterization result of Fisher and Jerome [1].

THEOREM 6. *Suppose* (I) *and* (II) *are satisfied. Then there is a fundamental interval* $J = [x_{\lambda_1}, x_{\lambda_2+n_0}]$, *for some* $1 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant m - n_0$ *with* $\sum_{i=\lambda_1}^{\lambda_2+n_0} k_i \geqslant n + 1$ *such that any two solutions of* (4.2) *agree on J. Moreover, if* $g \in S(U^*)$, *then* $|Lg| = \alpha_0^*$ a.e. *on J. If* g^* *is chosen as in Theorem 5, then* g^* *is unique in* $S_m(U^*)$. *Moreover,* g^* *has the property that* $|Lg^*|$ *is equivalent to a step function on* $[a, b]$ *with discontinuities restricted to* $x_2, \ldots, x_{m-1}$ *and, on* (x_i, x_{i+1}), $i = 1, \ldots, m - 1$, Lg^* *is equivalent to a step function with at most* $n - 1$ *discontinuities on each such interval.*

This theorem establishes existence of what is reasonably termed a "piecewise perfect-spline" solution of the extremal problem. Further, the characterization determines an interval of uniqueness and bounds the number of knots of a solution on each subinterval (x_i, x_{i+1}). It will be clear in the following discussion that our Theorem 1 assures existence of a piecewise perfect spline solution of (4.2) even without assumptions (I) and (II). The more precise characterizations of Theorem 6, however, rely on these assumptions.

Building on these results, Fisher and Jerome [2] construct global perfect spline solutions of (4.2) for the particular case $L = D^n$. It is this specialization, in particular, that we relax. They prove the following result.

THEOREM 7. *There is a perfect spline solution g to the extremal problem* (4.2) *when* $L = D^n$, *provided the functionals* L_{ij} *satisfy hypothesis* (II). *g has the property that* $D^n g = \pm\alpha_0^*$ *except at a finite number of points of discontinuity of* $D^n g$, *which cannot exceed n in number on* (x_i, x_{i+1}) *for each* $i = 1, \ldots, m - 1$.

We prove the following generalization.

THEOREM 8. *There is a perfect spline solution g to the extremal problem* (4.2). *g has the property that* $Lg = \pm\alpha_0^*$ *except at a finite number of points of discontinuity of Lg. If assumption* (I) *is satisfied, i.e., if* L^* *possesses Property T on* $[a, b]$, *then the number of knots of g in each interval* (x_i, x_{i+1}), *for* $1 \leqslant i \leqslant m-1$, *need not exceed n.* (*Further, if assumption* (II) *is satisfied by the functionals* L_{ij} *then there is a fundamental subinterval* $J = [x_{\lambda_1}, x_{\lambda_2+n_0}]$ *of uniqueness of g as described in Theorem* 6.)

Proof. Let f_0 be any solution of (4.1) and (4.2). Its existence is assured by Theorem 6 of Fisher and Jerome. Fix i between 1 and $m-1$ and consider the restriction of f_0 to (x_i, x_{i+1}). Define $x_0^{(\nu+1)} = f_0^{(\nu)}(x_i)$ and $x_1^{(\nu+1)} = f_0^{(\nu)}(x_{i+1})$. Consistent with the notations of Section 2, define

$$U = \{f \in W^{n,\infty}: f^{(\nu)}(x_i) = x_0^{(\nu+1)} \text{ and } f^{(\nu)}(x_{i+1}) = x_1^{(\nu+1)} \text{ for } \nu = 0,..., n-1\}.$$

Also define $\alpha_0 = \inf_{f\in U} \| Lf\|_{L_\infty(x_i,x_{i+1})}$. Certainly $\alpha_0 \leqslant \alpha_0^*$, since f_0 is a function in U with $\| Lf\|_\infty = \alpha_0^*$. If $\alpha_0 = \alpha_0^*$, then Theorem 1 assures that f_0 is the unique solution in U of $\| Lf_0 \| = \alpha_0$ and f_0 is a perfect spline on (x_i, x_{i+1}). If $\alpha_0 < \alpha_0^*$, then Theorem 4 assures the existence of a perfect spline g_i in U, which is not unique, satisfying $| Lg_i | = \alpha_0^*$ on (x_i, x_{i+1}) except at knots of g_i.

We can thus construct a perfect spline g_i on each subinterval (x_i, x_{i+1}) so that $\| Lg_i \|_{L_\infty(x_i,x_{i+1})} = \alpha_0^*$. Define g on $[a, b]$ by $g(t) = g_i(t)$ for $x_i \leqslant t \leqslant x_{i+1}$ and $1 \leqslant i \leqslant m-1$. Since $f_0 \in C^{n-1}[a, b]$, also $g \in C^{n-1}[a, b]$. Further, g is a perfect spline on $[a, b]$ and $| Lg | = \alpha_0^*$ except at knots of g. This establishes the existence statement of Theorem 8.

When assumption (I) is imposed, the limit on the number of knots that g must have on each subinterval (x_i, x_{i+1}) follows from Theorem 4.

The last statement of the theorem is from the characterization of [2, Theorem 6].

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